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Partial Abelian Semigroups

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Orthomodular lattices and posets, orthoalgebras, and D-posets are all examples of partial Abelian semigroups. So, too, are the event structures of test spaces. The passage from an algebraic test space to its logic (an orthoalgebra) is an instance of a general construction involving a partial Abelian semigroup L and a distinguished subset $M \subseteq L$ such that perspectivity with respect to M is a congruence on L. The quotient of L by such a congruence is always a cancellative, unital PAS, and every such PAS arises canonically as such a quotient.

INTRODUCTION

Recently, certain ordered sets equipped with a partial addition have received much attention as generalizations of "quantum logics," i.e., of orthomodular lattices and posets. These include orthoalgebras (Foulis *et al.*, 1992) and the so-called *D-posets* introduced by Kôpka and Chovanec (1994). The purpose of this note is to point out that the study of such objects gains much in clarity when cast in terms of a more general theory of partial Abelian semigroups. For instance, the representation of an orthoalgebra as the logic of a manual (or algebraic test space) is seen to be a special case of a very general construction. In particular, one sees easily how to obtain a representation theory for D-posets that as "logics" of certain collections of integer-valued functions.

In what follows, we shall omit most of the proofs, which are quite straightforward. A more detailed account will appear elsewhere.

1. PARTIAL ABELIAN SEMIGROUPS, ORTHOALGEBRAS, AND D-ALGEBRAS

By a *partial Abelian semigroup* (PAS) we mean a structure $(L, \perp, \oplus)$, where $\perp$ is a binary relation on L and $\oplus$ is a partially defined binary operation

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with domain $\perp$ satisfying

$$p \oplus q = q \oplus p \tag{1}$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r) \tag{2}$$

(These identities are to be understood as asserting that if the term on either side is defined, so is that on the other, and the two are equal.)

We say that a PAS L is *cancelative* iff for all $a, b, c \in L$, $a \perp c \perp b$ and $a \oplus c = b \oplus c$ entail $a = b$. A *zero* in L is an element 0 such that $p \perp 0$ for all $p \in L$ and $p \oplus 0 = 0$. If L is cancelative, it has at most one zero. One can always adjoin a zero formally; we shall therefore assume henceforth that every PAS (*and, in particular, every Abelian semigroup*) *possesses a zero.* We shall say that L is *positive* iff for all $p, q \in L$, $p \oplus q = 0$ only if $p = q = 0$.

Any PAS carries a natural preordering $p \leq q \Leftrightarrow \exists r\, p \oplus r = q$.

Lemma 1. If L is positive and cancelative, $\leq$ is a partial ordering. If $\leq$ is a partial ordering, L is positive.

The following examples are of particular importance.

1. Let $\mathscr{A}$ be a collection of sets, and let $E = E(\mathscr{A})$ denote the set of all subsets of elements of $\mathscr{A}$. For $a, b \in L$, set $a \perp b$ iff $a \cup b \in E$ and $a \cap b = \emptyset$, in which case define $a \oplus b = a \cup b$. Then E is a cancelative, positive PAS.

2. Let $\mathscr{F}$ be a collection of nonnegative integer-valued functions on a set X, and let $E(\mathscr{F})$ denote the set of all functions $f\colon X \to \mathbb{Z}_+$ such that for some $e \in \mathscr{F}$, $f \leq e$. Set $f \perp g$ iff $f + g \in E$, in which case define $f \oplus g = f + g$. Note that if all the functions in $\mathscr{F}$ are $\{0, 1\}$-valued, we recover Example 1.

A *unit* in a PAS L is an element $1 \in L$ such that for every $a \in L$, there exists at least one $b \in L$ such that $a \oplus b = 1$. A *unital* PAS is a PAS with a distinguished unit 1. Evidently, an element $1 \in L$ is a unit iff $a \leq 1$ for every $a \in L$. If $\leq$ is a partial ordering, L has at most one unit—in particular, if L is cancelative and positive, 1 is unique.

Let L be a cancelative, unital PAS with unit 1. We call L a *D-algebra* iff for all $p \in L$

$$p \perp 1 \text{ exists} \Rightarrow p = 0 \tag{3}$$

L is an *orthoalgebra* iff for every $p \in L$,

$$p \perp p \text{ exists} \Rightarrow p = 0 \tag{4}$$

Our terminology is intended as a hybrid: What we call a D-algebra coincides with what Foulis and Bennett (1994) call an *effect algebra*, which in turn is the same thing as a D-poset in the sense of Kôpka and Chovanec (1994).

It is easy to see that a cancelative PAS is a D-algebra iff it is positive and bounded as a poset. Thus, examples of D-algebras are plentiful; indeed, any cancelative, positive PAS L is infested with D-algebras: For any $e \in L$ and any $p, q \le e$, define $p \perp_e q$ iff $p \oplus q \le e$. Then $([0, e], \perp_e, \oplus, 0, e)$ is a D-algebra.

1.1. Homomorphisms and Congruences

If L and M are two PASs, a function f: $L \to M$ is a *homomorphism* iff $f(0) = 0\, p \perp q \Rightarrow f(p) \perp f(q)$ and $f(p \oplus q) = f(p) \oplus f(q)$. A homomorphism is *faithful* iff $f(a) \perp f(b) \Rightarrow a \perp b$.

Note that an isomorphism (i.e., a homomorphism with a two-sided inverse) is the same thing as a bijective, faithful homomorphism. The trivial homomorphism f: $S \to \{0\}$ is faithful iff S is a semigroup, i.e., iff $a \perp b$ for all $a, b \in s$.

A subset M of a PAS L is a *sub-PAS* iff it is closed under existing sums. If f: $L \to S$ is faithful, $f(L) \subseteq S$ is a sub-PAS of S.

A *congruence* on a PAS S is an equivalence relation $\simeq$ on S such that for all $a, b, c \in S$, $a \simeq b$ and $a \perp c$ imply $b \perp c$ and $a \oplus c \simeq b \oplus c$.

Let $\simeq$ be a congruence on L and denote by $[a]$ the equivalence class of $a \in L$ under $\simeq$. The partial operation $[a] \oplus [b] = [a \oplus b]$, defined for pairs $a \perp b$, is well defined, and makes $L/\simeq$ into a PAS. The map $[\cdot]$: $L \to L/\simeq$ is a faithful homomorphism. Conversely, if f: $L \to M$ is a faithful surjective homomorphism, the relation $a \simeq b \Leftrightarrow f(a) = f(b)$ is a congruence, and L is canonically isomorphic to $L/\simeq$.

Not every equivalence relation on a PAS L generates a congruence. Indeed, the universal relation $L \times L$ is a congruence iff the homomorphism f: $L \to \{0\}$ is faithful, in which case L is a semigroup.

1.2. Summable Sets and Functions

In dealing with a partial binary operation, even one that is commutative and associative, the notion of summability is a bit delicate. Things do work as one would hope, but to establish this requires some care. Here, we merely remark that for $a_1, \ldots, a_n \in L$, the sum $a_1 \oplus \cdots \oplus a_n$ exists iff any of the iterated sums, e.g., $(\ldots (a_1 \oplus a_2) \oplus \ldots) \oplus a_n$, exist, and is independent of the ordering. We shall call a finitely nonzero function f: $I \to L$ *summable* iff the sum $\oplus_{i \in I} f(i)$ exists. A subset A of L is summable iff the inclusion map i_A: $A \to L$ is summable. One can prove that if f, g: $I \to L$ are summable, then $f + g$ is summable iff $\oplus f \perp \oplus g$, and in this case, $\oplus(f + g) = (\oplus f) \oplus (\oplus g.)$

For any $a \in L$, we define a partial function $n \mapsto na$ from $\mathbb{Z}_+$ to S by induction: Define $1a = a$. If na has been defined and $na \perp a$, then set $(n + 1)a = na \oplus a$. Let $\rho(a)$ be the greatest $n \in \mathbb{N}$ for which na is defined, if any, and set $\rho(a) = \infty$ otherwise. We call $\rho(a)$ the *rank* of $a \in S$. Note that if $n + k \leq \rho(a)$, then $(n + k)a = na \oplus ka$.

A *summable function* on L is a finitely nonzero function $f: L \to \mathbb{Z}_+$ such that $\oplus f := \oplus_{a \in S} f(a)a$ exists. We denote the collection of summable functions on L by $\mathfrak{S}(L)$. If $f \in \mathfrak{S}(L)$ and $g: L \to \mathbb{Z}_+$ with $g(a) \leq f(a)$ for all $a \in S$, then $g \in \mathfrak{S}(L)$; hence, $\mathfrak{S}(L)$ is a PAS under the restricted addition $f \oplus g = f + g$, provided this is again in $\mathfrak{S}(L)$.

One has the following generalized associative law:

Lemma 2. For all f, $g \in \mathfrak{S}(L)$, $f \perp g$ iff $\oplus f \perp \oplus g$, and in this case, $\oplus f \oplus \oplus g = \oplus(f + g)$.

Note that the substance of this is that the map $f \mapsto \oplus f$ is a faithful surjective homomorphism from $\mathfrak{S}(L)$ into L. Thus: *Every PAS is the faithful homomorphic image of its PAS of summable functions.* In particular, every PAS is the faithful homomorphic image of a cancelative, positive PAS.

2. ALGEBRAIC SETS

There is a standard representation theory for orthoalgebras in terms of so-called *manuals* or *algebraic test-spaces*. In this section, we introduce a generalization of this notion, and in particular, produce a representation for D-algebras in terms of manual-like collections of functions.

Let L be a PAS and $M \subseteq L$. Say that a, $b \in L$ are *complementary* relative to M, writing $a\ c\ b$, iff $a \perp b$ and $a \oplus b \in M$. We say that a and b are *perspective* relative to M, writing $a \sim b$, iff there exists $x \in L$ such that $a\ c\ x\ c\ b$. Note that $a \sim 0$ iff $a\ c\ b$ for some $b \in M$, and that if a, $b \in M$, then $a \sim b$.

A subset $M \subseteq L$ is *algebraic* iff the relation $\sim$ is a congruence. If $\sim_M$ is the congruence on L induced by an algebraic subset $M \subseteq L$, we write L/M for $L/\sim_M$ and $[\cdot]_M$ for the canonical surjection $L \to L/M$.

If M is algebraic, then $a \sim a$ for every $a \in L$; hence, there is some $c \in L$ with $a \oplus c \in M$. Thus, an algebraic set must be dominating with respect to the preorder $\leq$, i.e., for every $a \in L$ there is some $b \in M$ with $a \leq b$.

By way of example, let $\mathscr{A}$ be any collection of sets and let $L = E(\mathscr{A})$. Then $M = \mathscr{A}$ is algebraic in L iff $\mathscr{A}$ is algebraic in the sense of Foulis and Bennett (1994); in that case, L/M is the logic of $\mathscr{A}$.

Lemma 3. Let M be an algebraic subset of L. Then L/M is cancelative and unital.

Lemma 4. Let $\phi: L \to L'$ be a faithful surjective homomorphism. If $L \subseteq M$ is algebraic, then $\phi^{-1}(M) \subseteq L$ is algebraic and $L/\phi^{-1}(M) \simeq L'/M$.

Using the preceding lemmas, it is not hard to prove the following:

Proposition 1. Let L be any unital PAS and let M denote the set of all $f \in \mathfrak{S}(L)$ such that $\oplus f = 1$. Then M is dominating and algebraic, and $\Delta(L) := \mathfrak{S}(L)/M$ is universal among faithful cancellative images of L in that any faithful homomorphism $\phi: L \to L'$, L' a cancellative PAS, factors uniquely through $\Delta(L)$.

A similar result is available if L is not unital. We omit the details.

From Proposition 1 we obtain the following generalization of the representation theorem for orthoalgebras:

Corollary. Let L be a cancellative, unital PAS and let $M \subseteq \mathfrak{S}(L)$ be as in Proposition 2. Then $L \simeq \mathfrak{S}(L)/M$.

Notice that a D-algebra is an orthoalgebra iff the rank function $\rho(a)$ is $\{0, 1\}$-valued; in this case, $\mathfrak{S}(L)$ consists likewise of $\{0, 1\}$-valued functions, i.e., of the characteristic functions of subsets of L. Hence $E(L) = \mathfrak{S}(L)$, and M is simply the manual of finite orthopartitions of 1 in L. Thus, we recover the standard representation theory of orthoalgebras as logics of manuals as a special case of Proposition 1. We should also mention that the representation theory developed here is closely related to one given by Dvurecenskij and Pulmannová (1994) in terms of so-called *D-test spaces*. For a comparison of the two approaches, see Pulmannová and Wilce (1994).

D-Posets and effect algebras have been put forward as models of "fuzzy" or "unsharp" quantum logics. Since a standard model of a fuzzy set is a $[0, 1]$-valued function, the following construction seems particularly intriguing.

Let $L = [0, 1]$ (understood as a D-algebra), and let $\mathcal{A}$ be a manual of sets with outcome set X. Let $\mathcal{M}$ denote the collection of all characteristic functions χ_E with $E \in \mathcal{A}$. For each $f: X \to [0, 1]$, let $A_f = \{x \mid 0 < f(x) < 1\}$ and $S_f = \{x \mid f(x) > 0\} = A_f \cup f^{-1}(1)$. Then $E(\mathcal{M})$ is the collection of all functions f with $S_f \in E(\mathcal{A})$. Note that for all $A \in E(\mathcal{A})$, $\chi_A \in E(\mathcal{M})$. Now let $f, g, h, k \in E(\mathcal{M})$ with $f + g = \chi_E$, $g + h = \chi_F$, and $h + k = \chi_G$ for some $E, F, G \in \mathcal{A}$. Then $A_f = A_g = A_h = A_k = A \subseteq E \cap F \cap G$ and $f + k \equiv 1$ on A. Also note that S_f c $g^{-1}(1)$ c S_h c $k^{-1}(1)$. Since $\mathcal{A}$ is a manual, S_f c $k^{-1}(1)$, whence $S_f \cup k^{-1}(1) = H \in \mathcal{A}$. Since $A_f = A$, we have $H = f^{-1}(1) \cup A \cup k^{-1}(1)$; it follows that $f + k = \chi_H$. Hence, $\mathcal{M}$ is algebraic in $E(\mathcal{M})$, whence $\Pi(\mathcal{M}) = E(\mathcal{M})/\sim$ is a cancelative, unital PAS by Lemma 3.

In fact, we claim $\Pi(\mathcal{M})$ is a D-algebra. It suffices to show that $\Pi(\mathcal{M})$ is positive. Suppose $f, g \in E(\mathcal{M})$. If $f \oplus g \sim_M 0$, then $f \oplus g \oplus \chi_E = \chi_F$ for some E, $F \in \mathscr{A}$. It follows that $E \subseteq F$. Since $\mathscr{A}$ is algebraic, this entails $E = F$, whence $f \oplus g$ is identically 0 on X. Hence, f, $g = 0$. Thus, $\Pi(\mathscr{M})$ is a D-algebra, as claimed.

Note that for A, $B \in E(\mathscr{A})$, $\chi_A \sim \chi_B$ iff $A \sim B$ in $E(\mathscr{A})$ with respect to $\mathscr{A}$—i.e., $\Pi(\mathscr{M})$ contains the orthoalgebra $\Pi(\mathscr{A})$ as a sub-D-algebra.

It should be noted that this construction works perfectly well with *any* D-algebra in place of [0, 1]. More generally, if X is any set, L is any PAS, and $M \subseteq L^X$, we may construct the PAS $E(M)$ consisting of functions $g \in L^X$ with $g \leq f$ for some $f \in M$. Let us agree to call M an *L-manual* iff M is algebraic in $E(M)$ and $E(M)/M$ is a D-algebra. Thus, the example above is a [0, 1]-manual, while a {0, 1}-manual is simply a manual of sets in the ordinary sense. If L is cancelative and positive, then for any L-manual the quotient $\Pi(M) := E(M)/M$ will be a D-algebra.

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